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Probability Theory and Applications (MA208)
Problem Sheet - 5

Functions of Random Variables

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1. Suppose that X is uniformly distributed over $(-1, 1)$. Let $Y = 4 - X^2$. Find the pdf of Y, say $g(y)$, and sketch it. Also verify that $g(y)$ is a pdf.

2. Suppose that X is uniformly distributed over $(1, 3)$. Obtain the pdf of the following random variables:

   (a) $Y = 3X + 4$
   (b) $Z = e^X$.

   Verify in each case that the function obtained is a pdf. Sketch the pdf.

3. Suppose that the continuous random variable X has pdf $f(x) = e^{-x}, x > 0$. Find the pdf of the following random variables:

   (a) $Y = X^3$
   (b) $Z = 3/(X+1)^2$.

4. Suppose that the discrete random variable X assumes the values $1, 2$, and 3 with equal probability. Find the probability distribution of $Y = 2X + 3$.

5. Suppose that X is uniformly distributed over the interval $(0, 1)$. Find the pdf of the following random variables:

   (a) $Y = X^2 + 1$
   (b) $Z = 1/(X+1)$.

6. Suppose that X is uniformly distributed over $(-1, 1)$. Find the pdf of the following random variables:

   (a) $Y = \sin(\pi/2)X$
   (b) $Z = \cos(\pi/2)X$
   (c) $W = |X|$.

7. Suppose that the radius of a sphere is a continuous random variable. (Due to inaccuracies of the manufacturing process, the radii of different spheres may be different.) Suppose that the radius R has pdf $f(r) = 6r(1-r), 0 < r < 1$. Find the pdf of the volume V and the surface area S of the sphere.

8. A fluctuating electric current I may be considered as a uniformly distributed random variable over the interval $(9, 11)$. If this current flows through a 2-ohm resistor, find the pdf of the power $P = 2I^2$.

9. The speed of a molecule in a uniform gas at equilibrium is a random variable V whose pdf is given by

   $$f(v) = av^2 e^{-bv^2}, \quad v > 0$$

   where $b = m/2kT$ and k, T, and m denote Boltzman's constant, the absolute temperature, and the mass of the molecule, respectively.

(a) Evaluate the constant a (in terms of b). [Hint: Use the fact that $\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$ and integrate by parts.]

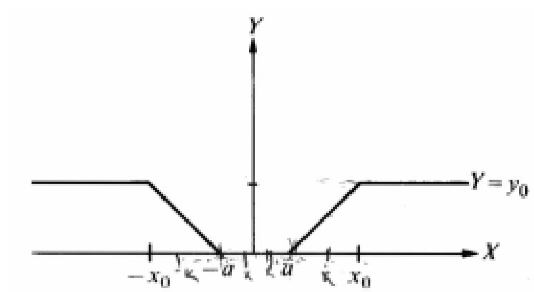
(b) Derive the distribution of the random variable $W = mV^2/2$, which represents the kinetic energy of the molecule.

10. A random voltage X is uniformly distributed over the interval $(-k, k)$. If Y is the input of a nonlinear device with the characteristics shown in the following figure. Find the probability distribution of Y in the following three cases:

    (a) $k < a$

    (b) $a < k < x_0$

    (c) $k > x_0$.



Note: The probability distribution of Y is an example of a mixed distribution. Y assumes the value zero with a *positive* probability and also assumes all values in certain intervals.

11. The radiant energy (in $Btu/hr/ft^2$) is given as the following function of temperature T (in degree fahrenheit): $E = 0.173(T/100)^4$. Suppose that the temperature T is considered to be a continuous random variable with pdf

$$f(t) = 200t^{-2}, \quad 40 \le t \le 50,$$
$$= 0, \quad \text{elsewhere.}$$

Find the pdf of the radiant energy E.

12. To measure air velocities, a tube (known as Pitot static tube) is used which enables one to measure differential pressure. This differential pressure is given by $P = (1/2)dV^2$, where d is the density of the air and V is the wind speed (mph). If V is a random variable uniformly distributed over $(10, 20)$, find the pdf of P.

13. Suppose that $P(X \le 0.29) = 0.75$, where X is a continuous random variable with some distribution defined over $(0, 1)$. If $Y = 1 - X$, determine k so that $P(Y \le k) = 0.25$.

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